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Highly Accelerated Laminar Flow at Moderately Large Reynolds Number

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Jointly self-similar equations for first- and the second-order boundary-layer flows of an incompressible fluid with uniform freestream are studied for highly accelerated flows (Falkner-Skan pressure gradient parameter $\beta \to \infty$). The first four terms in an asymptotic expansion as $\beta \to \infty$, for first-order and each of the second-order effects (due to longitudinal curvature, transverse curvature, and displacement) are evaluated. The results for skin friction so obtained give a good agreement with known exact results for $\beta = 1$. For smaller values of β the convergence of the series are accelerated by Eulerization, which show very good agreement with the exact results not only for β equal to zero but for negative values as well.

Nomenclature

A = displacement thickness due to first-order boundary layer defined by Eq. (9)
D = function defined by Eq. (8)
f = first-order stream function $\psi_1/\sqrt{(2\xi)}$
F = second-order stream function $\psi_2/\sqrt{(2\xi)}$
j = a number defined as zero for two-dimensional flow and unity for axisymmetric flow
K = longitudinal surface curvature of the body
k_l, k_t = longitudinal and transverse curvature parameters defined by Eqs. (6) and (7)
n = coordinate normal to body
N = stretched normal coordinate defined as $= R^{1/2}n$
r = radius of the body
R = characteristic Reynold number of the flow
s = coordinate along the body
U_1, U_2 = first- and second-order outer flow velocities in s direction
Z = Eulerized variable defined as $1/(1+\beta)$
ξ, η = Gortler variables defined by Eq. (3)
α = defined as $A/(\varepsilon)^{1/2}$
β = Falkner-Skan pressure gradient parameter defined as $= 2\xi/U_1\, dU_1/d\xi$
θ = angle between the axis and the tangent to the meridian curve at any point
ε = $1/\beta$
τ_1, τ_2 = first- and second-order skin friction defined by Eq. (10)

Subscripts
η = total differentiation with respect to variable η

Superscripts
$'$ = total differentiation with respect to variable ξ
d = displacement speed
l = longitudinal curvature
t = transverse curvature

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1. Introduction

THE problem of highly accelerated flows has attracted the attention of many workers in the recent years. Under certain conditions (high acceleration, low Reynolds number effects associated with high altitude flights, etc.) laminar boundary layers are found over larger portions of the surface. Even the turbulent-boundary layers, under large pressure gradients, have been found to revert towards laminar boundary layers (Launder,[1] Badrinarynan and Ramjee[2]). Further the problem is also of interest in wind-tunnel contraction, turbine nozzle cascades, rocket nozzle, etc.

The main aim of the present work is to study the effect of large accelerations at moderately large values of Reynolds number. The oncoming stream is assumed uniform and fluid to be incompressible. Van Dyke[3] has proposed a theory for analyzing the effects of moderately large Reynolds number using the technique of matched asymptotic expansions. These effects are of order $R^{-1/2}$ compared to classical boundary-layer theory (first-order). For the present case of uniform freestream these effects arise due to longitudinal curvature, transverse curvature, and displacement speed. Using the similarity variables these equations are transformed into self-similar equations by Afzal and Oberai.[4] The first-order equation is the well known, nonlinear Falkner-Skan equation with a pressure gradient parameter β. The second-order boundary-layer equation is linear, hence each of the effects can be studied separately.

In the present work these first- and second-order boundary-layer equations in jointly self-similar flow situations are analyzed for large values of β. For first-order and each of the second-order effects (longitudinal curvature, transverse curvature, and displacement effects), the first four terms in the asymptotic expansions as $\beta \to \infty$ are computed. The present results show good agreement with the known exact results for $1 \leq \beta < \infty$. However, for $\beta < 1$ the convergence of the present series solutions become poor due to a limited radius of convergence. The radius of convergence of the series is extended by Eulerization. The Eulerized results show good agreement with the exact results not only for β equal to zero but for negative values also.

2. First- and Second-Order Boundary-Layer Equations

A systematic formulation of the second-order boundary-layer equtions has been given by Van Dyke,[3] who has studied the inner and outer limits of the Navier-Stokes equations and matched them in the overlap domain. The general structure of the self-similarity for the second-order equations has been analyzed by Afzal and Oberai.[4] The final form of the jointly self-similar equations obtained by them are as follows.

2.1 First-Order Boundary-Layer Equation

$$f_{\eta\eta\eta}+ff_{\eta\eta}+\beta(1-f_\eta{}^2)=0 \tag{1}$$

$$f(o)=0=f_\eta(o); \quad f_\eta(\infty)=1 \tag{2a, b, c}$$

This is well known Falkner-Skan equation. Here $(2\xi)^{1/2} f(\eta)$ is the first-order stream function and β the pressure gradient parameter. The variables ξ and η are defined as

$$\xi=\int_0^s r^{2j}U_1(s,o)\,ds, \quad \eta=r^jU_1(s,o)N/\sqrt{(2\xi)} \tag{3a, b}$$

The suffix η denotes total derivative with respect to η.

2.2 Second-Order Boundary-Layer Equation

$$F_{\eta\eta\eta}+fF_{\eta\eta}-2\beta f_\eta F_\eta+f_{\eta\eta}F=k_l\left[-\eta f_{\eta\eta\eta}+\frac{\beta-1}{\beta+1}(f_{\eta\eta}+ff_\eta)+\frac{2\beta}{\beta+1}(\beta\eta+A)\right]+k_t\left[-\eta(f_{\eta\eta\eta}+2\beta)+f_{\eta\eta}+ff_\eta\right]-2D\beta \tag{4}$$

$$F(o)=0=F_\eta(o) \tag{5a, b}$$

$$F_\eta(\eta)\sim -k_l\eta+k_t\eta+D \quad \text{as} \quad \eta\to\infty \tag{5c}$$

In the above equations the terms proportional to k_l arise because of longitudinal curvature, k_t due to transverse curvature and D due to displacement. These parameters are defined as

$$k_l=\sqrt{(2\xi)}K/[r^jU_1(s,o)] \tag{6}$$

$$k_t=\sqrt{(2\xi)}j\cos\theta/[r^{j+1}U_1(s,o)] \tag{7}$$

$$D=U_2(s,o)/U_1(s,o) \tag{8}$$

and should be constants for joint self-similar flows. Further $\sqrt{(2\xi)}F(\eta)$ is the second-order stream function and

$$A=\lim_{\eta\to\infty}(\eta-f) \tag{9}$$

The skin friction τ_1 and τ_2 for first- and second-order boundary-layer problems are

$$\tau_1=\frac{r^jU_1{}^2(s,o)}{\sqrt{(2\xi)}}f_{\eta\eta}(o), \quad \tau_2=\frac{r^jU_1{}^2(s,o)}{\sqrt{(2\xi)}}F_{\eta\eta}(o) \tag{10a, b}$$

3. Analysis at Large β

3.1 First-Order Boundary-Layer Equation

With $\varepsilon=1/\beta$ Eqs. (1) and (2) can be rewritten as

$$1-f_\eta{}^2+\varepsilon(f_{\eta\eta\eta}+ff_{\eta\eta})=0 \tag{11}$$

$$f(o)=0=f_\eta(o); \quad f_\eta(\infty)=1 \tag{12a, b, c}$$

For small values of ε, a straight forward outer asymptotic expansion for f is

$$f(\eta)=f_0(\eta)+\varepsilon f_1(\eta)+\varepsilon^2 f_2(\eta)+\dots \tag{13}$$

Solutions to equations for f_0, f_1 etc. satisfying the boundary conditions at infinity are $f_0=\eta-A$, $f_1=f_2=0$, which shows that the outer expansion is trivial.

To get insight of the flow near the wall, we make an order of magnitude analysis. Introducing the new variables

$$f(\eta)=B(\beta)g(\zeta), \quad \eta=C(\beta)\zeta \tag{14a, b}$$

in Eq. (11) gives

$$\varepsilon B/C^3\,g'''+\varepsilon B^2/C^2\,gg''+[1-(B^2/C^2)g'^2]=0 \tag{15}$$

Here prime denotes the total derivative with respect to ζ. As $\varepsilon\to 0$ (high accelerating flow) it is the pressure gradient which is to be balanced by viscous terms, in the neighbourhood of the wall, to get

$$B=C=\varepsilon^{1/2} \tag{16}$$

The final form of the variables (14) are

$$f(\eta)=g(\zeta)\varepsilon^{1/2}, \quad \eta=\zeta\varepsilon^{1/2} \tag{17a, b}$$

and Eq. (15) and its boundary conditions reduce to

$$g'''+1-g'^2+\varepsilon gg''=0 \tag{18}$$

$$g(o)=0=g'(o); \quad g'(\infty)=1 \tag{19a, b, c}$$

The asymptotic expansion of $g(\zeta)$ is hopefully in ε

$$g=\sum_{m=0} g_m\varepsilon^m \tag{20}$$

Substituting expansion (20) in Eq. (18) and its boundary conditions (19), the following equations are obtained

$$g_m'''+\delta_{0m}-\sum_{n=0}^{n=m} g_n'g_{m-n}'+\sum_{n=0}^{n=m} g_{n-1}g_{m-n}''=0 \tag{21}$$

$$g_m(o)=0=g_m'(o); \quad g_m'(\infty)=\delta_{0m} \qquad m=0,1,2,3\dots \tag{22a, b, c}$$

where δ_{0m} is the Kronecker delta.

The first equation (corresponding to $m=0$) in Eq. (21) is similar to that of sink flow whose solution is[5]

$$g_0=\zeta+2(3)^{1/2}-3(2)^{1/2}\tanh[\zeta/(2)^{1/2}+\tanh^{-1}(2/3)^{1/2}] \tag{23}$$

3.2 Second-Order Problem

Following the first-order analysis, it is easily shown that outer expansion for second-order problem is again trivial and we can study the problem in terms of the new variables (17). Equation (4) along with boundary conditions (5) is linear, hence each of the second-order effect can be studied separately and then finally be superposed. This is achieved by

$$F(\eta)=k_lF^{(l)}+k_tF^{(t)}+DF^{(d)} \tag{24}$$

a) Longitudinal curvature

The momentum equation for longitudinal curvature problem is

$$F_{\eta\eta\eta}{}^{(l)}+fF_{\eta\eta}{}^{(l)}-2\beta f_\eta F_\eta{}^{(l)}+f_{\eta\eta}F^{(l)}=-\eta f_{\eta\eta\eta}+\frac{\beta-1}{\beta+1}(f_{\eta\eta}+ff_\eta)+\frac{2\beta}{\beta+1}(\beta\eta+A) \tag{25}$$

Boundary conditions are

$$F^{(l)}(o)=0=F_\eta{}^{(l)}(o); \quad F_\eta{}^{(l)}(\eta)=-\eta \quad \text{as} \quad \eta\to\infty \tag{26a, b, c}$$

Introducing variables (17) in Eq. (25) and boundary conditions (26) we get (dropping superscript "l" for the convenience)

$$F''' - 2g'F' + \varepsilon(gF'' + g''F) = -\varepsilon\zeta g''' + \frac{1-\varepsilon}{1+\varepsilon}(\varepsilon g'' + \varepsilon^2 gg') + \frac{2\varepsilon}{(1+\varepsilon)}(\zeta + \varepsilon\alpha) \tag{27}$$

$$F(o) = 0 = F'(o); \quad F'(\zeta) = -\varepsilon\zeta \quad \text{as} \quad \zeta \to \infty \tag{28}$$

$$\alpha = \frac{A}{\varepsilon^{1/2}} = \lim_{\zeta \to \infty}[\zeta - g(\zeta)] \tag{29}$$

From the first-order analysis the expansion for α is

$$\alpha = \alpha_0 + \varepsilon\alpha_1 + \varepsilon^2\alpha_2 + \varepsilon^3\alpha_3 + \ldots \tag{30}$$

where

$$\alpha_0 = \lim_{\zeta \to \infty}[\zeta - g_0(\zeta)] \quad \text{and} \quad \alpha_n = -g_n(\infty), \quad n = 1, 2, 3, \ldots \tag{31}$$

Assuming asymptotic expansion for $F(\zeta)$ as

$$F(\zeta) = \varepsilon F_0 + \varepsilon^2 F_1 + \varepsilon^3 F_2 + \varepsilon^4 F_3 + \ldots \tag{32}$$

and substitution of expansion (32) in Eq. (27) and boundary-conditions (28) gives

$$F_m''' - 2\sum_{n=0}^{m} g_{m-n}'F_n + \sum_{n=0}^{m}(g_{m-n}F_{n-1}'' + g_{m-n}''F_{n-1}) = g_m''' + (-1)^m 2\zeta + \sum_{n=0}^{m} 2(-1)^{m-1}\alpha_{n-1}g_m'' - \sum_{n=0}^{m} 2(-1)^m g_{m-n} - \sum_{n=0}^{m} g_{n-1}g_{m-n}' + 2\sum_{p=0}^{p=m}\sum_{n=0}^{n=p} g_{n-1}g_{m-1}'(-1)^{p-n},$$

$$m = 0, 1, 2, 3, \ldots \tag{33}$$

$$F_m(o) = 0 = F_m'(o); \quad F_m'(\zeta) = -\zeta\delta_{0m} \quad \text{as} \quad \zeta \to \infty \tag{34a, b, c}$$

b) Transverse curvature

Governing equation for transverse curvature problem is

$$F_{\eta\eta\eta}^{(t)} + fF_{\eta\eta}^{(t)} - 2\beta f_\eta F_\eta^{(t)} + f_{\eta\eta}F^{(t)} = -\eta(f_{\eta\eta\eta} + 2\beta) + f_{\eta\eta} + ff_\eta \tag{35}$$

and the corresponding boundary conditions are

$$F^{(t)}(o) = 0 = F_\eta^{(t)}(o); \quad F_\eta^{(t)}(\eta) = \eta \quad \text{as} \quad \eta \to \infty \tag{36a, b, c}$$

Introducing the variables (17) in the above equation we get (dropping the superscript "t" for convenience)

$$F''' - 2g'F' + \varepsilon(gF'' + g''F) = -\varepsilon\zeta(2 + g''') + \varepsilon g'' + \varepsilon^2 gg' \tag{37}$$

$$F(o) = 0 = F'(o); \quad F'(\zeta) = \varepsilon\zeta \quad \text{as} \quad \zeta \to \infty \tag{38a, b, c}$$

The asymptotic expansion

$$F(\zeta) = \varepsilon F_0 + \varepsilon^2 F_1 + \varepsilon^3 F_2 + \varepsilon^4 F_3 + \ldots \tag{39}$$

upon substitution in Eqs. (35) and (36), gives the various equations as

$$F_m''' - 2\sum_{n=0}^{n=m} g_{m-n}'F_n' + \sum_{n=0}^{n=m}(g_{m-n}F_{n-1}'' + g_{m-n}''F_{n-1}) = -2(\delta_{0m} + g_m''') + g_m'' + \sum_{n=0}^{n=m} g_{m-n}g_{n-1} \tag{40}$$

$$F_m(o) = 0 = F_m'(o); \quad F_m'(\zeta) = \delta_{0m}\zeta \quad \text{as} \quad \zeta \to \infty$$

$$m = 0, 1, 2, 3, \ldots \tag{41a, b, c}$$

c) Displacement speed

The governing equations for displacement effect are

$$F_{\eta\eta\eta}^{(d)} + fF_{\eta\eta}^{(d)} - 2\beta f_\eta F_\eta^{(d)} + f_{\eta\eta}F^{(d)} = -2\beta \tag{42}$$

$$F^{(d)}(o) = 0 = F_\eta^{(d)}(o); \quad F_\eta^{(d)}(\infty) = 1 \tag{43a, b, c}$$

Introducing the variables (17) in the Eqs. (42) and (43) (dropping the superscript "d" for convenience)

$$F''' - 2g'F' + \varepsilon(gF'' + g''F) = -2\varepsilon^{1/2} \tag{44}$$

$$F(o) = 0 = F'(o); \quad F'(\infty) = \varepsilon^{1/2} \tag{45a, b, c}$$

Introducing the asymptotic expansion

$$F = \varepsilon^{1/2}[F_0 + \varepsilon F_1 + \varepsilon^2 F_2 + \varepsilon^3 F_3 + \ldots] \tag{46}$$

in the Eqs. (44) and (45) the various equations are

$$F_m''' - 2\sum_{n=0}^{n=m} g_{m-n}'F_n' + \sum_{n=0}^{n=m}(g_{m-n}F_{n-1}'' + g_{m-n}''F_{n-1}) = -2\delta_{0m} \tag{47}$$

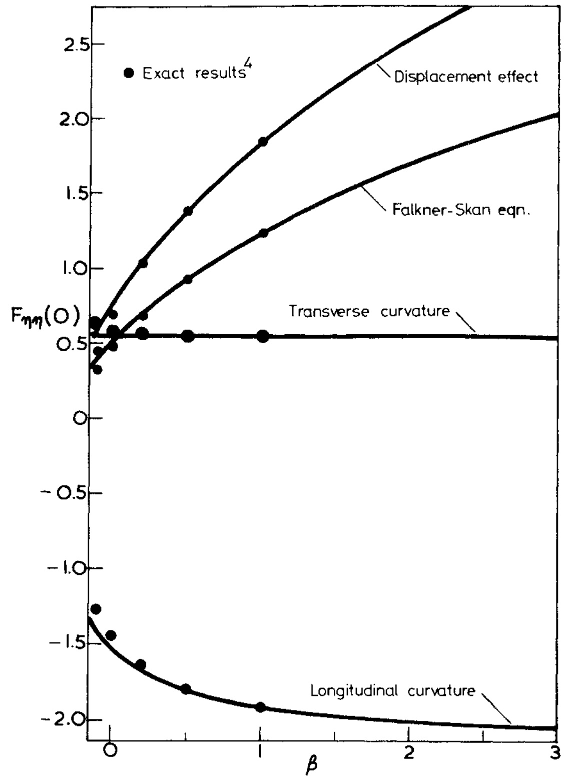

Fig. 1 Comparison of present results for first- and second-order boundary-layer equations with exact results.

$$F_m(o) = 0 = F_m'(o); \quad F_m'(\infty) = \delta_{0m} \quad m = 0, 1, 2, 3, \ldots \tag{48a, b, c}$$

The closed form solutions for Eq. (47) in terms of first-order boundary-layer variables is

$$F_m = 1/2(\zeta g_m' + g_m) \tag{49}$$

The value of the second derivative at the wall is

$$F_m''(o) = \tfrac{3}{2}g_m''(o) \tag{50}$$

Results and Discussion

First four equations for first-order and each of the second-order effects (longitudinal curvature, transverse curvature, and displacement speed) are integrated numerically by Runge-Kutta-Gill method (step size $\Delta\eta = 0.05$) on IBM 7044 computer at IIT Kanpur. The results for skin friction are

$$f_{\eta\eta}(o) = 1.1547\beta^{1/2} + 0.0746\beta^{-1/2} + 0.00509\beta^{-3/2} - 0.00182\beta^{-5/2} + \ldots \tag{51}$$

$$F_{\eta\eta}^{(l)}(o) = -2.1575 + 0.3744\beta^{-1} - 0.1974\beta^{-2} + 0.0998\beta^{-3} + \ldots \tag{52}$$

$$F_{\eta\eta}^{(t)}(o) = 0.5394 + 0.0126\beta^{-1} - 0.0082\beta^{-2} + 0.0024\beta^{-3} + \ldots \tag{53}$$

$$F_{\eta\eta}^{(d)}(o) = 1.7320\beta^{1/2} + 0.1119\beta^{-1/2} + 0.00764\beta^{-3/2} - 0.00274\beta^{-5/2} + \ldots \tag{54}$$

The preceding series solutions have been obtained for $\beta \to \infty$,

Table 1 Comparison of results, Eqs. (51–54) for skin friction at $\beta = 1$

Effect	Present results	Exact results	% error
First order	1.2326	1.2326	0
Longitudinal curvature	−1.8908	−1.9136	1.5
Transverse curvature	0.5462	0.5466	0
Displacement speed	1.8489	1.8489	0

Table 2 Comparison of Eulerized results Eqs. (55–58) for skin friction at $\beta = 0$

Effect	Partial sum I	II	III	IV	Exact results	% error
First order	1.1547	0.6520	0.5501	0.5117	0.4696	9
Longitudinal curvature	−2.1575	−1.7831	−1.6062	−1.5288	−1.4470	5
Transverse curvature	0.5394	0.5520	0.5564	0.5550	0.5714	3
Displacement speed	1.7320	0.9780	0.8252	0.7676	0.7044	12

but they converge even for $\beta = 1$. Comparison of the results from above series with the exact results of Afzal and Oberai[4] for $\beta = 1$, shown in Table 1, are extremely good. As β further decreases, the convergence of the series become poor, because of limited radius of convergence, and become singular at $\beta = 0$. It is well known that such asymptotic series do contain much information and very useful estimates of the sum of the slowly convergent or even divergent series can often be obtained by use of various transformations which accelerate the convergence. Several interesting examples of such transformations are given by Van Dyke,[6] and one of the widely used being the process called Eulerization (see also Meksyn[7]). The Eulerization of series (48)–(51) give

$$f_{\eta\eta}(o) = 1.1547Z^{-1/2} - 0.5027Z^{1/2} - 0.1019Z^{3/2} - 0.0384Z^{5/2} + \ldots \quad (55)$$

$$F_{\eta\eta}^{(l)}(o) = -2.1575 + 0.3744Z + 0.1769Z^2 + 0.0794Z^3 + \ldots \quad (56)$$

$$F_{\eta\eta}^{(t)}(o) = 0.5394 + 0.0126Z + 0.0044Z^2 - 0.0014Z^3 + \ldots \quad (57)$$

$$F_{\eta\eta}^{(d)}(o) = 1.7320Z^{-1/2} - 0.7540Z^{1/2} - 0.1528Z^{3/2} - 0.0576Z^{5/2} + \ldots \quad (58)$$

where

$$Z = 1/(1+\beta)$$

As $\beta \to 0$, $Z \to 1$, the successive partial sums of the series are shown in Table 2. The last partial sum converges very close to exact results of Afzal and Oberai.[4] The comparison of present Eulerized results for various other values of β are shown in Fig. 1.

From these results it is clear that how the study of limiting case of $\beta \to \infty$ is of use even for β not only zero, but for negative values also, provided a large number of terms in the expansion are computed and the series are properly Eulerized.

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